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ON THE CONTINUITY OF PROJECTIONS AND A GENERALIZED GRAM-SCHMIDT PROCESS

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Abstract

Let Ω be an open connected subset of the complex plane $\mathbb{C}$ and let T be a bounded linear operator on a Hilbert space H. For λ in Ω let $P_{N(T-\lambda I)}$ be the orthogonal projection onto the null-space of T-λI. We discuss the necessary and sufficient conditions for the map $\lambda \to P_{N(T-\lambda I)}$ to be continuous on Ω. A generalized Gram-Schmidt process is also given.

Introduction

For a connected open subset Ω of the plane and n a positive integer, let $\mathcal{B}_n(\Omega)$ denote the class of operators T defined on the Hilbert space H which satisfy

(a) $\Omega \subseteq \sigma(T)$, the spectrum of T,

(b) ran $(T-\omega) = H$ for ω in Ω,

(c) $\bigvee_{\omega \in \Omega} \ker(T-\omega) = H$, and

(d) $\dim \ker (T-\omega) = n$ for ω in Ω.

The space $\mathcal{B}_n(\Omega)$ has been introduced by Cowen and Douglas [1]. Curto and Salinas [2] also study this class of operators from a different point of view. While studying these operators, Curto and Salinas [2] come across conditions for which $P_{\mathrm{ran}(T-\omega)}$ projection onto the range of T-ω is a continuous function of ω. In particular they prove the equivalence of (a) and (c) of Theorem 2.1.

In the present article we give a simpler proof of their result using elementary techniques. We also prove more. We show that the analytic properties such as continuity of the projections have algebraic implications (part (d) and (e) of Theorem 2.1). The proof of the equivalence of (a) and (d) of Theorem 2.1 takes a bit of work. However, it is interesting in the sense that it shows something of an analytic nature can be described entirely algebraically.

Finally, we consider idempotents $E_1, \ldots, E_n$ that are pairwise disjoint ($E_iE_j = 0$, $i \neq j$) and decompose the space ($E_1 + \ldots + E_n = I$) and use a kind of Gram-Schmidt process to replace them by orthogonal projections $P_1, \ldots, P_n$ having the same property as $E_1, \ldots, E_n$ ($P_iP_j = 0$, $i \neq j$, and $P_1 + \ldots + P_n = I$). Moreover, there is a common invertible operator S which satisfies $SE_i = P_iS$ for $i = 1, \ldots, n$. The existence of such an invertible operator is useful in orthogonalizing a given basis of a vector space.

1. Preliminaries. Let H be a complex separable Hilbert space and let B(H) denote the class of all bounded linear operators defined on H. For T in B(H) the range and null space of T are denoted by R(T) and N(T) respectively. We also set

$$m(T) = \inf\{\|Tx\| : x \perp N(T), \|x\| = 1\}.$$

Note that $m(T) > 0$ if and only if R(T) is closed. If M is a (closed) subspace of H then the orthogonal

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projection onto M is **denoted by** P_M.

2. Continuity of the projections. In this section we would like to prove the following:

2.1. Theorem. Let Ω be a domain in $\mathbb{C}$ such that $m(T-\lambda) > 0$ for all λ in Ω. Then the following conditions are equivalent.

(a) $\lambda \to P_{N(T-\lambda)}$ is continuous at λ_0.

(b) $\lambda \to m(T-\lambda)$ is continuous at λ_0.

(c) $\lambda \to P_{R(T-\lambda)}$ is continuous at λ_0.

(d) For every $n \geq 1$ we have $N(T-\lambda_0)^n \subset R(T-\lambda_0)$.

(e)For every $n \geq 1$ we have $N(T-\lambda_0) \quad R(T-\lambda_0)^n$

Proof. (a) $\Rightarrow$ (b).

If $\lambda \to P_{N(T-\lambda)}$ is cotinuous at λ_0 then there exists $\delta > 0$ such that $\| P_{N(T-\lambda)} - P_{N(T-\lambda_0)} \| < 1$ whenever $| \lambda - \lambda_0 | < \delta$. It follows that $I - (P_{N(T-\lambda)} - P_{N(T-\lambda_0)})$ and $I - (P_{N(T-\lambda_0)} - P_{N(T-\lambda)})$ are invertible. Hence

$$N(T-\lambda_0) + N(T-\lambda)^{\perp} = H$$

$$N(T-\lambda) + N(T-\lambda_0)^{\perp} = H$$

Suppose x is a unit vector, $x \perp N(T-\lambda)$; then $x = y + z$, $y \in N(T-\lambda)$ and $z \perp N(T-\lambda_0)$. Note that $z \neq 0$. Now

$$\| (T-\lambda)x \| = \| (T-\lambda)z \| \geq \| (T-\lambda_0)z \| - |\lambda - \lambda_0| \, \| z \|$$
$$\geq m(T-\lambda_0) \|z\| - | \lambda - \lambda_0 | \, \|z\|.$$

Since $z = x - y$ we have

$\|z\|^2 = \|x\|^2 + \|y\|^2 = 1 + \|y\|^2 \geq 1$.

Therefore

$$\| (T-\lambda)x \| \geq m(T-\lambda_0) - |\lambda - \lambda_0|.$$

Taking the infimum over all such x we have

$$m(T-\lambda) \geq m(T-\lambda_0) - | \lambda - \lambda_0 |.$$

or

$$m(T-\lambda_0) - m(T-\lambda) \leq | \lambda - \lambda_0 |.$$

Similarly

$$m(T-\lambda) - m(T-\lambda_0) \leq | \lambda - \lambda_0 |.$$

Hence

$$| m(T-\lambda) - m(T-\lambda_0) | \leq | \lambda - \lambda_0 |.$$

We have now established (b).

(b) $\Rightarrow$ (c)

We know by [3,Chap. 4] that

$\| P_{R(T-\lambda)} - P_{R(T-\lambda_0)} \| \leq \max \{c,d\}$, where

$$c = \sup \left\{ \text{dist}(y, R(T-\lambda_0)) : y \in R(T-\lambda), \|y\| = 1 \right\} \text{ and}$$
$$d = \sup \left\{ \text{dist}(y, R(T-\lambda)) : y \in R(T-\lambda_0), \|y\| = 1 \right\}.$$

Every vector is $R(T-\lambda)$ is of the form $(T-\lambda)z$, $z \perp N(T-\lambda)$. Let

$$y = \frac{(T-\lambda)z}{\|(T-\lambda)z\|} \text{ and } x = \frac{(T-\lambda_0)z}{\|(T-\lambda)z\|}.$$

Then

$$| \lambda - \lambda_0 | \, \|z\| = \| (T-\lambda)z - (T-\lambda_0)z \| = \| (T-\lambda)z \| \, \| y - x \|$$
$$\geq \| (T-\lambda)z \| \, \text{dist}(y, R(T-\lambda_0))$$
$$\geq m(T-\lambda) \, \| z \| \, \text{dist}(y, R(T-\lambda_0)).$$

Taking the supermum over all y in $R(T-\lambda)$, $\|y\| = 1$, we obtain that $c \cdot m(T-\lambda) \leq | \lambda - \lambda_0 |$. Interchanging the role of λ and λ_0 we obtain

$$\| P_{R(T-\lambda)} - P_{R(T-\lambda_0)} \| \leq | \lambda - \lambda_0 | \left[\frac{1}{m(T-\lambda)} + \frac{1}{m(T-\lambda_0)} \right].$$

The proof of this part is also complete.

(c) $\Rightarrow$ (d)

Let $x \in N(T-\lambda_0)^n$. For $\lambda \neq \lambda_0$ we write

$$0 = (T-\lambda_0)^n x = \left[(T-\lambda) + (\lambda - \lambda_0) \right]^n x$$
$$= \sum_{k=0}^{n} \binom{n}{k} (T-\lambda)^k (\lambda - \lambda_0)^{n-k} x.$$

It follows that

$$(\lambda - \lambda_0)^n x = -\sum_{k=1}^{n} \binom{n}{k} (T-\lambda)^k (\lambda - \lambda_0)^{n-k} x.$$

Therefore $x \in R(T-\lambda)$ for $\lambda \neq \lambda_0$. Write

$$x = P_{R(T-\lambda)} x = P_{R(T-\lambda)} x - P_{R(T-\lambda_0)} x + P_{R(T-\lambda_0)} x.$$

Letting $\lambda \to \lambda_0$ and using the continuity of $\lambda \to P_{R(T-\lambda)}$ we obtain that $x = P_{R(T-\lambda_0)} x$, from which we conclude that $x \in R(T-\lambda_0)$.

The proof that (d) and (e) are equivalent is left as an exercise.

We now show that (d) $\Rightarrow$ (a). This is done by use of two lemmas.

2.2 Lemma. Let E,F be idempotents ($E^2 = E$, $F^2 = F$) with ranges M, N respectively, If P,Q are orthogonal projections onto M, N respectively then

$$\| P - Q \| \leq \| E - F \|.$$

The proof can be found in [3] which is quite complicated; however, we present a simple proof in case $N \subset M$; this is done in the hope that a simpler proof can be found for the general case by reduction to this particular case.

So assume $N \subset M$ and decompose the space into $N \oplus (M \ominus N) \oplus M^{\perp}$. With respect to this decomposition

$$E = \begin{bmatrix} I & 0 & A \\ 0 & I & B \\ 0 & 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} I & C & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$E - F = \begin{bmatrix} 0 & -C & A-D \\ 0 & I & B \\ 0 & 0 & 0 \end{bmatrix}.$$

If we apply E-F to the vector $z=0 \oplus y \oplus 0$ we obtain

$$\| (E-F)z \|^2 = \| Cy \|^2 + \| y \|^2 \geq \| y \|^2 = \| z \|^2.$$

Hence $\|E-F\| \geq 1$. But P-Q is the orthogonal projection onto $M \ominus N$, so it has norm 1.

Therefore $\|E-F\| \geq \|P-Q\|$ and the proof is complete.

In passing we feel that it might be interesting to know the relationship between an idempotent E and the orthogonal projection with range R(E). This is done next.

2.3 Proposition. For an idempotent E, the orthogonal projection P onto the range of E is given by

$$P = EE^*(I - (E - E^*)^2)^{-1}.$$

Proof. Decompose the space into $R(E) \oplus R(E)^{\perp}$. With respect to this decomposition

$$E = \begin{bmatrix} I & A \\ 0 & 0 \end{bmatrix},$$

then

$$E^* = \begin{bmatrix} I & 0 \\ A^* & 0 \end{bmatrix}, \qquad E - E^* = \begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}$$

$$(E - E^*)^2 = \begin{bmatrix} -AA^* & 0 \\ 0 & -A^*A \end{bmatrix}.$$

But $I - (E - E^*)^2 = \begin{bmatrix} I+AA^* & 0 \\ 0 & I+A^*A \end{bmatrix}$ is invertible.

Also

$EE^* = \begin{bmatrix} I+AA^* & 0 \\ 0 & 0 \end{bmatrix}$. A simple computation now completes the proof.

2.4. Lemma. Let $T \in B(H)$ have closed range and satisfy

$$N(T^n) \subset R(T) \text{ for all } n \geq 1.$$

If there is an operator S such that TST = T then $SN(T^n) \subset N(T^{n+1})$ for all $n \geq 1$. In particular $S^jN(T) \subset N(T^{j+1})$ for all $j \geq 0$. and $S^jN(T) \subset R(T)$ for $J \geq 0$.

Proof. If $y \in N(T^n)$ then since $N(T^n) \subset R(T)$ we have $y = Tx$ for some x. Now $T^{n+1}Sy = T^nTSTx = T^nTx = T^ny = 0$.

To complete the proof of Theorem 2.1 we need to prove (d) ⇒ (a). This will be done next.

(d) ⇒ (a)

Without loss of generality we may assume that $\lambda_0 = 0$. Note that since R(T) is closed $\tilde{T}: N(T)^{\perp} \to R(T)$, the restriction of T to $N(T)^{\perp}$, is invertible. We define

$$S = \begin{cases} \tilde{T}^{-1} & \text{on } R(T) \\ 0 & \text{on } R(T)^{\perp} \end{cases}.$$

It is easy to see that $ST = I - P_{N(T)}$, $TS = P_{R(T)}$, $TST = T$ and $STS = S$.

For $|\lambda| < \frac{1}{\|S\|}$ we define

$S_\lambda = (I - \lambda S)^{-1}S = S(I - \lambda S)^{-1}$. We will show that $(T-\lambda I)S_\lambda(T-\lambda I) = T - \lambda I$ and $S_\lambda(T-\lambda I)S_\lambda = S_\lambda$ for $|\lambda| < \frac{1}{\|S\|}$.

To prove these let $|\lambda| < \frac{1}{\|S\|}$. Then

$$\begin{aligned} S_\lambda(T-\lambda I)S_\lambda &= (I-\lambda S)^{-1}S(T-\lambda I)S(T-\lambda I)^{-1} \\ &= (I-\lambda S)^{-1}(STS-\lambda S^2)(I-\lambda S)^{-1} \\ &= (I-\lambda S)^{-1}(S-\lambda S^2)(I-\lambda S)^{-1} \\ &= S_\lambda. \end{aligned}$$

We also have

$$\begin{aligned} (T-\lambda I)S_\lambda(T-\lambda I) &= (T-\lambda I)(I-\lambda S)^{-1}S(T-\lambda I) \\ &= (T-\lambda I) - (T-\lambda I)(I-\lambda S)^{-1}P_{N(T)} \end{aligned}$$

But

$$(T - \lambda I)(I-\lambda S)^{-1}P_{N(T)} = (T-\lambda I)\sum_{j=0}^{\infty} \lambda^j S^j P_{N(T)}$$

$$= \sum_{j=0}^{\infty} \lambda^j P_{R(T)} S^{j-1} P_{N(T)} - \sum_{j=0}^{\infty} \lambda^{j+1} S^j P_{N(T)}$$

$$= \sum_{j=0}^{\infty} \lambda^j S^{j-1} P_{N(T)} - \sum_{j=1}^{\infty} \lambda^j S^{j-1} P_{N(T)} = 0 .$$

We now define $E_\lambda = (T-\lambda I) S_\lambda$ and $F_\lambda = S_\lambda (T-\lambda I)$. Using the equalities we just proved we have $E_\lambda (T-\lambda I) = T-\lambda I$ and $(T-\lambda I) F_\lambda = (T-\lambda I)$. It is easy to see that the operators E_λ and F_λ are idempotents with $R(E_\lambda) = R(T-\lambda I)$ and $N(F_\lambda) = N(T-\lambda I)$. By Lemma 2.2 $\| P_{N(T-\lambda I)} - P_{N(T-\mu I)} \| \leq \| (I - F_\lambda) - (I - F_\mu) \| = \| F_\lambda - F_\mu \|$.

Since $\lambda \to F_\lambda$ is analytic in the ball $| \lambda | < \frac{1}{\|S\|}$ the proof follows immediately.

3. Generalized Gram-Schmidt Process. In Lemma 2.2 and Proposition 2.3 we have stated a relationship between idempotents and corresponding orthogonal projections having the same range. In this section we would like to follow the same vein of ideas and state a generalized Gram-Schmidt process.

Let $E_1, E_2, \ldots, E_n$ be in B(H) such that $E_i^2 = E_i$ (i=1,..., n), $E_i E_j = 0$ (i≠j) and $E_1 + \ldots + E_n = I$. For i=1, ...,n set $W_i = R(E_i)$ then clearly $H = W_1 \oplus \ldots \oplus W_n$. This sum might not be an orthogonal one. To ameliorate the situation we let Q_i be the orthogonal projection onto $R(E_1 + \ldots + E_i) = W_1 \oplus \ldots \oplus W_i = V_i$ and we set $P_1 = Q_1$, $P_i = Q_i - Q_{i-1}$ for i=2.,...,n. Actually, P_i is the orthogonal projection onto $V_i \theta V_{i-1}$ for $i \geq 2$. Moreover $R(P_1 + \ldots + P_i) = R(E_1 + \ldots + E_i)$ and $H = H_1 \oplus \ldots \oplus H_n$ where $H_i = R(P_i)$. This sum being an orthogonal direct sum. We also have $P_i P_j = 0$, i≠j, and $P_1 + \ldots + P_n = I$.

We now summarize what we have done as follows. For each n-tuple of idempotents $(E_1, \ldots, E_n)$ with $E_i E_j = 0$, i≠j, $E_1 + \ldots + E_n = I$ we found an n-tuple of orthogonal projections $(P_1, \ldots, P_n)$ having the same properties. We now show that the two n-tuples are similar with one similarity simultaneously intertwining each such pair.

3.1. Theorem. The operator $S = \sum_{i=1}^{n} P_i E_i$ is invertible and $SE_i = P_i S$ for i=1,...,n. Proof. It suffices to show that S is invertible. We first show that $P_i E_j = E_i P_j = 0$ if j < i. To see this note that if j=1 then $E_i P_1 = E_i E_1 P_1 = 0$ for i > 1. Now assume that j > 1.

Then $E_i P_j = E_i Q_j - E_i Q_{j-1} = 0$ since $R(Q_j)$ and $R(Q_{j-1})$ are subsets of $V_j = W_1 \oplus \ldots \oplus W_j$ and $E_i = 0$ on V_j (1 < j < i). In the same way we can show that $P_i E_j = 0$ for j < i.

Now

$$I = \left[\sum_{i=1}^{n} P_i \right] \left[\sum_{j=1}^{n} E_j \right] = \sum_{i=1}^{n} P_i E_i + \sum_{i<j} P_i E_j , \text{ so } S = I - \sum_{i<j} P_i E_j .$$

Let $T_k = I + \sum_{k<m} P_k E_m$. Then $ST_1 T_2 \ldots T_n = I$ as is easily seen by computing this product. Therefore S is invertible.

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